

Generalized modified gravity models: the stability issue

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Abstract

A brief introduction on the issue of stability in generalized modified gravity is presented and the dynamical system methods are used in the investigation of the stability of spatially flat homogeneous cosmologies within a large class of generalized modified gravity models in the presence of a relativistic matter-radiation fluid.

1 Introduction

To start with, we recall that recent cosmological data support the fact that there is a good evidence for a late accelerated expansion of the observable universe, apparently due to the presence of an effective positive and small cosmological constant of unknown origin. This is known as dark energy issue (see for example [1]).

Among several existing explanation, the so called modified gravity models are possible realizations of dark energy (for a recent review and alternative approaches see [2, 3, 4]), which may offer a quite natural geometrical approach again in the spirit of the original Einstein theory of gravitation. In fact, the main idea underlying these approaches to dark energy puzzle is quite simple and consists in adding to the gravitational Einstein-Hilbert action other gravitational terms which may dominate the cosmological evolution during the very early or the very late universe epochs, but in such a way that General Relativity remains valid at intermediate epochs and also at non cosmological scales.

The Λ -CDM model is the simplest possibility but, it is worth investigating more general modifications, possible motivations run from quantum corrections to string models. We shall first consider the simpler modification of the kind $F(R) = R + f(R)$, and then discuss the related generalizations. Models based on $F(R)$ are not new and they have been used in the past by many authors, for example as models for inflation, $f(R) = aR^2$ [5]. Recently their interest in cosmology was triggered by the model $f(R) = -\mu^4/R$, proposed in order to describe the current acceleration of the observable universe [6, 7]. For incomplete list of references, see [8].

It is important to stress that these $F(R)$ models are conformally equivalent to Einstein's gravity, coupled with a self-interacting scalar field, Einstein frame formulation. We will consider only the Jordan frame, in which the dynamics of gravity is described by $F(R)$ with minimally coupled matter. Observations are typically interpreted in this Jordan frame.

We would like also to mention the so called viable $F(R)$ models, which have recently been proposed [9, 10, 11, 12, 13, 14], with the aim to describe the current acceleration with a suitable choice of $F(R) = R + f(R)$, but also to be compatible with local stringent gravitational tests of Einstein gravity $F(R) = R$. The main idea is the so called disappearing of cosmological constant for low curvature, and mimicking the Λ -CDM model for high curvature. Thus, the requirements are:

- $f(R) \rightarrow 0$, $R \rightarrow 0$, compatibility with local tests;
- $f(R) \rightarrow -2\Lambda_0$, $R \rightarrow +\infty$, description of current acceleration;
- Local stability of the matter.

As a illustration, we recall a recent example of viable model [15]

$$f(R) = -\alpha \left(\tanh \left(\frac{b(R - R_0)}{2} \right) + \tanh \left(\frac{bR_0}{2} \right) \right)$$

where R_0 , and Λ_0 are suitable constants. Its advantages are a better formulation in the Einstein frame and a generalization that may also include the inflation era.

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2 The de Sitter stability issue for $F(R)$ models

The stability of the de Sitter solution, relevant for Dark energy, may be investigated in the $F(R)$ models in several ways. We limit ourselves to the following three approaches:

- i. perturbation of the equations of motion in the Jordan frame;
- ii. one-loop gravity calculation around de Sitter background;
- iii. dynamical system approach in FRW space-time.

We shall briefly discuss first two approaches, and then concentrate to the third one, since it is the only that can be easily extended to more general modified gravitational models, which is the main issue of this short review.

i. Stability of $F(R)$ model in the Jordan frame

The starting point is the trace of the equations of motion, which is trivial in Einstein gravity $R = -\kappa^2 T$, but, for a general $F(R)$ model, reads

$$3\nabla^2 f'(R) - 2f(R) + Rf'(R) - R = \kappa^2 T.$$

The new non trivial extra degree of freedom is the scalaron: $1 + f'(R) = e^{-\chi}$. Requiring $R = R_0 = CST$, one has de Sitter existence condition in vacuum

$$R_0 + 2f(R_0) - R_0 f'(R_0) = 0.$$

Perturbing around dS: $R = R_0 + \delta R$, with $\delta R = -\frac{1+f'(R_0)}{f''(R_0)}\delta\chi$, one arrives at the scalaron perturbation equation

$$\nabla^2 \delta\chi - M^2 \delta\chi = -\frac{\kappa^2}{6(1+f'(R_0))}T.$$

One may read off the scalaron effective mass

$$M^2 \equiv \frac{1}{3} \left(\frac{1+f'(R_0)}{f''(R_0)} - R_0 \right).$$

Thus, if $M^2 > 0$, one has stability of the dS solution and the related condition reads

$$\frac{1+f'(R_0)}{R_0 f''(R_0)} > 1.$$

If $M^2 < 0$, there is a tachyon and instability. Furthermore, one may show that M^2 has to be very large in order to pass both the local and the astronomical tests and $1 + f'(R) > 0$, in order to have a positive effective Newton constant. The same result has been obtained within a different more general perturbation approach in [16].

ii. One-loop $F(R)$ quantum gravity partition function

Here we present the generalization to the modified gravitational case of the study of Fradkin and Tseytlin [17], concerning Einstein gravity on dS space. One works in the Euclidean path integral formulation, with dS existence condition $2F_0 = R_0 F_0$, assumed to be satisfied. The small fluctuations around this dS instanton may be written as

$$g_{ij} = g_{0,ij} + h_{ij}, \quad g^{ij} = g_0^{ij} - h^{ij} + h^{ik} h_k^j + \mathcal{O}(h^3), \quad h = g_0^{ij} h_{ij}.$$

Making use of the standard expansion of the tensor field h_{ij} in irreducible components, and making an expansion up to second order in all the fields, one arrives at a very complicated Lagrangian density L_2 , not reported here, describing Gaussian fluctuations around dS space. As usual, in order to quantise the model described by L_2 , one has to add gauge fixing and ghost contributions. Then, the computation of Euclidean one-loop partition function reduces to the computations of functional determinants. These functional determinants are divergent and may be regularized by the well known zeta-function

regularization. The evaluation requires a complicated calculation and, neglecting the so called multiplicative anomaly, potentially present in zeta-function regularized determinants (see [18]), one arrives at the one-loop effective action [19] (here written in the Landau gauge)

$$\begin{aligned}\Gamma_{on-shell} &= \frac{24\pi F_0}{GR_0^2} + \frac{1}{2} \log \det \left[\ell^2 \left(-\Delta_2 + \frac{R_0}{6} \right) \right] \\ &\quad - \frac{1}{2} \log \det \left[\ell^2 \left(-\Delta_1 - \frac{R_0}{4} \right) \right] \\ &\quad + \frac{1}{2} \log \det \left[\ell^2 \left(-\Delta_0 - \frac{R_0}{3} + \frac{2F_0}{3R_0 F_0''} \right) \right].\end{aligned}$$

The last term is absent in the Einstein theory. As a result, in the scalar sector one has an effective mass $M^2 = \frac{1}{3} \left(\frac{2F_0}{R_0 F_0''} - R_0 \right)$. Stability requires $M^2 > 0$, in agreement with the previous scalaron analysis, and with the inhomogeneous perturbation analysis [16].

The dynamical system approach

The main idea of such a method is to convert the generalized Einstein-Friedman equations in an autonomous system of first order differential equations and makes use of the theory of dynamical systems (see [20, 21, 22, 23, 24, 25] and references therein). We remind that the stability or instability issue is really relevant in cosmology. For example, in the Λ CDM model it ensures that no future singularities will be present in the solution. Within cosmological models, the stability or instability around a solution is of interest at early and also at late times.

The stability of de Sitter solutions has been discussed in several places, an incomplete list being [16, 26, 27, ?, 28, 29, 30, 31, 32, 33]. More complicated is the analysis of other critical points, associated with the presence of non vanishing matter and radiation. For example, in the paper [34], a non local model of modified gravity $F(R)$ has been investigated by means of this approach.

Here we shall extend to general modified gravity in the presence of matter the method given in Ref. [22], which permits to determine all critical points of a $F(R)$ model. Ordinary matter is important in reconstructing the expansion history of the Universe and probing the phenomenological relevance of the models (see for example the recent papers [22, 23, 24, 25], where the $F(R)$ case has been discussed in detail). Our generalisation consists in the extension of that method in order to include all possible geometrical invariants. This means that F could be a generic scalar function of curvature, Ricci and Riemann tensors.

There are some theoretical (quantum effects and string-inspired) motivations in order to investigate gravitational models depending on higher-order invariants. The “string-inspired” scalar-Gauss-Bonnet gravity case $F(R, G)$ has been suggested in Ref. [35] as a model for gravitational dark energy, while some time ago it has been proposed as a possible solution of the initial singularity problem [36]. The investigation of different regimes of cosmic acceleration in such gravity models has been carried out in Refs. [35, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46]. In particular, in [43] a first attempt to the study of the stability of such kind of models has been carried out using an approach based on quantum field theory.

The method we shall use in the present paper is based on a classical Lagrangian formalism, see, for example, [47, 48, 49], inspired by the paper [5], where quantum gravitational effects were considered for the first time. With regard to this, it is well known that one-loop and two-loops quantum effects induce higher derivative gravitational terms in the effective gravitational Lagrangian. Instability due to quadratic terms have been investigated in [50]. A particular case has been recently studied in [51] and general models depending on quadratic invariants have been investigated in [52, 53].

A stability analysis of nontrivial vacua in a general class of higher-derivative theories of gravitation has already been presented in [54]. Our approach is different from the one presented there since we are dealing with scalar quantities and moreover it is more general, since it is not restricted to the vacuum invariant submanifold.

Finally, it should be stressed that the stability studied here is the one with respect to homogeneous perturbations. For the $F(R)$ case, the stability criterion for homogeneous perturbations is equivalent to the inhomogeneous one [16]. In the following we will summarize the results obtained in [55].

2.1 The dynamical system approach: The general case

To start with, let us consider a Lagrangian density which is an arbitrary function of all algebraic invariants built up with the Riemann tensor of the FRW space-time we are dealing with, that is

$$\mathcal{L} = -\frac{1}{2\chi} F(R, P, Q, \dots) + \mathcal{L}_m, \quad (2.1)$$

where R is the scalar curvature, $P = R^{\mu\nu} R_{\mu\nu}$ and $Q = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$ are the two quadratic invariants and the dots means other independent algebraic invariants of higher order, and \mathcal{L}_\uparrow is the matter Lagrangian which depends on ρ and $p = p(\rho)$, the density and pressure of the matter.

For the sake of convenience we write the metric in the form

$$ds^2 = -e^{2n(t)} dt^2 + e^{2\alpha(t)} d\vec{x}^2, \quad N(t) = e^{n(t)}, \quad a(t) = e^{\alpha(t)}. \quad (2.2)$$

In this way $\dot{\alpha}(t) = H(t)$ is the Hubble parameter and a generic invariant geometrical quantity U has the form

$$U = e^{-2pn(t)} u(\dot{n}, \dot{\alpha}, \ddot{\alpha}) = e^{-2pn(t)} u(\dot{n}, H, \dot{H}) = H^{2p} e^{-2pn(t)} u(X), \quad (2.3)$$

where $2p$ is the dimension (in mass) of the invariant under consideration and $X = (\dot{H}/H^2 - \dot{n}/H)$ (see [55]). In particular one has

$$\begin{aligned} R &= 6H^2 e^{-2n} (2 + X), \\ P &= 12H^4 e^{-4n} (3 + 3X + X^2), \\ Q &= 12H^4 e^{-4n} (2 + 2X + X^2), \end{aligned} \quad (2.4)$$

Using this notation, the action reads

$$S = - \int d^3x \int dt L(n, \dot{n}, \alpha, \dot{\alpha}, \ddot{\alpha}) = \frac{1}{2\chi} \int d^3x \int dt e^{n+3\alpha} F(n, \dot{n}, \dot{\alpha}, \ddot{\alpha}) + S_m, \quad (2.5)$$

and the Lagrange equations corresponding to the two Lagrangian variables $n(t)$ and $\alpha(t)$ are given by

$$E_n = \frac{\partial L}{\partial n} - \frac{d}{dt} \frac{\partial L}{\partial \dot{n}} = 2\sqrt{-g} T_{00} g^{00} = 2\rho \sqrt{-g}, \quad (2.6)$$

$$E_\alpha = \frac{\partial L}{\partial \alpha} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\alpha}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{\alpha}} = \sqrt{-g} T_{ab} g^{ab} = -6p \sqrt{-g}. \quad (2.7)$$

It has to be noted that since $n(t)$ is a “gauge function”, which corresponds to the choice of the evolution parameter, Eqs. (2.6) and (2.7) are not independent and in fact they satisfy the differential equation

$$\frac{dE_n}{dt} = \dot{n}E_n + \dot{\alpha}E_\alpha, \quad (2.8)$$

which is equivalent to the conservation of energy-momentum tensor. Furthermore, we may use the gauge freedom and fix the cosmological time by means of the condition $N(t) = 1$, that is $n(t) = 0$. From now on it is understood that all quantities will be evaluated in such a gauge and so the parameter t corresponds to the standard cosmological time. In this way Eqs. (2.6) and (2.7) read

$$H\dot{F}_H - HF_H + F - \dot{H}F_{\dot{H}} + 3H^2F_{\dot{H}} = 2\rho, \quad (2.9)$$

$$\ddot{F}_H - \dot{F}_H + 6H\dot{F}_H - 3HF_H + 3F + 3\dot{H}F_{\dot{H}} + 9H^2F_{\dot{H}} = -6p. \quad (2.10)$$

The latter equations are the generalisation to arbitrary action of the well known Friedmann equations.

Now we shall replace Eqs. (2.9)-(2.10) with an autonomous system of first order differential equations. To this aim we first observe that in pure Einstein gravity, that is for $F = R$, (2.9)-(2.10) read

$$H^2F_{\dot{H}} = F_X = 2\rho \quad \implies \quad \Omega_\rho = 1, \quad (2.11)$$

$$(3H^2 + 2\dot{H})F_{\dot{H}} = (3 + 2X)F_X = -6p \quad \implies \quad \Omega_p = -1 - \frac{2}{3}X, \quad (2.12)$$

where we have introduced the dimensionless variables

$$\Omega_\rho = \frac{2\rho}{H^2 F_{\dot{H}}} = \frac{2\rho}{F_X}, \quad \Omega_p = \frac{2p}{H^2 F_{\dot{H}}} = \frac{2p}{F_X}, \quad (2.13)$$

which in this special case are given by the usual values $\Omega_\rho = \rho/3H^2$ and $\Omega_p = p/3H^2$. From Eqs. (2.11) and (2.12) it follows

$$w \equiv \frac{p}{\rho} = \frac{\Omega_\rho}{\Omega_p} = -1 - \frac{2}{3}X. \quad (2.14)$$

In the general case, Eqs. (2.9) and (2.10) have more terms with respect to (2.11) and (2.12) and it is quite natural to interpret them as corrections due to the presence of higher-order terms in the action. Then we define

$$\Omega_\rho^{eff} = \Omega_\rho + \Omega_\rho^{curv} = 1, \quad \Omega_p^{eff} = \Omega_p + \Omega_p^{curv} = -1 - \frac{2}{3}X, \quad (2.15)$$

$$w_{eff} \equiv \frac{\Omega_p^{eff}}{\Omega_\rho^{eff}} = -1 - \frac{2}{3}X, \quad (2.16)$$

where Ω_ρ^{curv} and Ω_p^{curv} are complicated expressions, which only depend on the function F . They can be derived from (2.9) and (2.10), but their explicit form is not necessary for our aims. The effective quantity w_{eff} is equal to the ratio between the effective density and the effective pressure and it could be negative even if one considers only ordinary matter. It is known that the current-measured value of w_{eff} is near -1 .

In order to get all critical points of the system now we follow the method, as described, for example, in [22]. First of all, we introduce the dimensionless variables

$$\Omega_\rho = \frac{2\rho}{H^2 F_{\dot{H}}} = \frac{2\rho}{F_X}, \quad \Omega_p = \frac{2p}{H^2 F_{\dot{H}}} = \frac{2p}{F_X}, \quad (2.17)$$

$$X = \frac{\dot{H}}{H^2}, \quad Y = \frac{F - HF_H}{H^2 F_{\dot{H}}} = \frac{F}{F_X} - X, \quad Z = \frac{\dot{F}_{\dot{H}} - F_H}{HF_{\dot{H}}} = \frac{F'_X}{F_X} - 2X - \xi, \quad (2.18)$$

where the prime means derivative with respect to α and the quantity

$$\xi = \xi(X, Y) = \frac{F_H}{HF_{\dot{H}}} = \frac{HF_H}{F_X}, \quad (2.19)$$

has to be considered as a function of the variables X and Y . In general it is a function of X and H , but this latter quantity can be expressed in terms of X and Y as a direct consequence of the definition of Y itself. Then we derive an autonomous system by taking the derivatives of such variables. From Eq. (2.9) we have the constraint

$$\Omega_\rho = Y + Z + 3 \quad \implies \quad \Omega_{curv} = -(Y + Z + 2), \quad (2.20)$$

which reduces to the standard one when F is linear in R (general relativity with cosmological constant).

Deriving the variables above by taking into account of (2.10) and (2.20) we get the system of first order differential equations

$$\begin{cases} X' = -2X^2 - \gamma X + \beta(Z + \xi) \\ Y' = -(2X + Z + \xi)Y - XZ \\ Z' = -3(1 + w)(Z + Y + 3) - (Z + \xi)(Z + 3) - X(Z + 6) \end{cases} \quad (2.21)$$

where $X' \equiv \frac{dX}{d\alpha} = \frac{1}{H} \frac{dX}{dt}$ (and so on) and for simplicity we have set $p = w\rho$, with constant w . For ordinary matter $0 \leq w \leq 1/3$ ($w = 0$ corresponds to dust, while $w = 1/3$ to pure radiation), but in principle one could also consider “exotic” matter with $w < 0$ or cosmological constant which corresponds to $w = -1$. We have also set

$$\beta = \beta(X, Y) = \frac{F_{\dot{H}\dot{H}}}{H^2 F_{\dot{H}\dot{H}}} = \frac{F_X}{F_{XX}}, \quad \gamma = \gamma(X, Y) = \frac{F_{H\dot{H}}}{HF_{\dot{H}\dot{H}}} = \frac{HF_{HX}}{F_{XX}} = \beta\xi_X + \xi. \quad (2.22)$$

It is understood that $F_{\dot{H}\dot{H}} \neq 0$ has been assumed. The quantity Ω_ρ at the critical points will be determined by means of Eq. (2.20).

The critical points are obtained by putting $X' = 0, Y' = 0, Z' = 0$ in the system (2.21). The number and the position of such points depends on the Lagrangian throughout the functions β, γ and ξ . In principle, given F one can derive all critical points, but in practice for a generic F the algebraic system could be very complicated and the solutions quite involved. We shall consider in detail some particular cases at the end of the Section.

As already said above, the critical points of (2.21) depends on ξ, β, γ , which in general are complicated functions of X and Y , then it is not possible to determine general solutions without to choose the model, nevertheless it is convenient to distinguish two distinct classes of solutions characterised by the values of $w \neq -1$ and $w = -1$. For the sake of completeness we consider $w \leq 1/3$ and so we write the solutions also for “exotic” matter, that is quintessence ($-1 < w < 0$) and phantom ($w < -1$). Of course, such solutions have to be dropped if one is only interested in ordinary matter/radiation. We have

- $w \neq -1$ — The critical points are the solutions of the system of three equations

$$\begin{cases} 2X^2 + \gamma X - \beta(Z + \xi) = 0 \\ (2X + Z + \xi)Y + XZ = 0 \\ 3(1 + w)(Z + Y + 3) + (Z + \xi)(Z + 3) + X(Z + 6) = 0 \end{cases} \quad w \neq -1, \quad (2.23)$$

where ξ, β, γ are functions of X, Y determined by Eqs. (2.19) and (2.22). The stability matrix has three eigenvalues and the point is stable if the real parts of all of them are negative.

The latter system has always the de Sitter solution $P_0 \equiv (X = 0, Y = 1, Z = -4)$, where $\Omega_\rho = 0$ and $w_{eff} = -1$. Note however that such a solution could exist also in the presence of matter, since the existence of P_0 critical point only implies that the critical value for Ω_ρ vanishes.

- $\Omega_\rho \neq 0, w = -1$ — The critical points are given by

$$\begin{cases} 2X^2 + \gamma X - \beta(Z + \xi) = 0 \\ (2X + Z + \xi)Y + XZ = 0 \\ (Z + \xi)(Z + 3) + X(Z + 6) = 0 \end{cases} \quad w = -1. \quad (2.24)$$

For this class of solutions, the non-singular stability matrix has three eigenvalues and the point is stable if the real parts of all of them are negative.

We see that there is at least one singular case (critical line) when $X = 0$ and $Z = -\xi = -4$. In fact in such a case Y or Ω_ρ are undetermined since

$$\Omega_\rho = Y + 3 - \xi(0, Y) = Y - 1 \quad \implies \quad Y = 1 + \Omega_\rho, \quad \Omega_\rho \text{ arbitrary}. \quad (2.25)$$

Such a solution can be seen as a generalisation of the de Sitter solution for a model with cosmological constant. The de Sitter critical point for the model $\tilde{F} = F - 2\Lambda$ reads $(X = 0, \tilde{Y} = 1, \tilde{Z} = -4)$. Such a solution follows from Eq. (2.25) if we choose $\rho_0 \equiv \Lambda$. In fact, on the critical point $(X = 0, Y = 1 + \Omega_\rho, Z = -4)$ (Eq. (2.25)) and from definitions (2.18) we get

$$\Omega_\rho = \frac{2\rho}{H^2 F_{\dot{H}}} = \frac{F}{H^2 F_{\dot{H}}} - 1 \quad \implies \quad \tilde{Y} \equiv \frac{\tilde{F}}{H^2 \tilde{F}_{\dot{H}}} = 1, \quad (2.26)$$

which corresponds to de Sitter critical point for \tilde{F} . Eq. (2.25) is more general than the case with pure cosmological constant since ρ is not necessary a constant, and for this special class of solutions $w_{eff} = -1$. Note also that the stability matrix has always a vanishing eigenvalue and the stability of the system is determined by the other two eigenvalues.

For some models, but just for technical reasons, it could be convenient to treat the cosmological constant as matter, using the previous identification we have done.

Explicit examples

In order to see how the method works, now we give explicit solutions for some models and, when possible we also study the stability of the critical points. We restrict our analysis to the values $0 \leq w \leq 1/3$ and to the special value $w = -1$, which corresponds to the pure cosmological constant, but in principle any negative value of w could be considered, even if this will be in contrast with the aim of modified gravity. In fact, modified gravity can generate an effective negative value of w without the use of phantom or quintessence.

It as to be stressed that in general, due to technical difficulties, one has to study the models by a numerical analysis. Only for some special cases one is able to find analytical results. Here we report the results for some models of the latter class in which the analytical analysis can be completely carried out. We also study more complicated models and for those we limit our analysis to the de Sitter solutions.

In the following we shall use the compact notation

$$P \equiv (X, Y, Z, \Omega_\rho, w_{eff}), \quad P_0 \equiv (0, 1, -4, 0, -1), \quad P_\Lambda = (0, 1 + \Omega_\Lambda, -4, \Omega_\Lambda, -1). \quad (2.27)$$

The latter is an additional critical point that we have for the choice $w = -1$ and can be seen as the de Sitter solution in the presence of cosmological constant.

$F = R - \mu^4/R$ — This is the well known model introduced in [6, 7] and discussed in [22]. For this model the system (2.21) with arbitrary w has six different solutions, but only two of them effectively correspond to physical critical points, if $0 \leq w \leq 1/3$. In principle there are other critical points for negative values of w (phantom or quintessence) and moreover there is also a particular solution for $w = -1$ which corresponds to the model with a cosmological constant Λ .

Solving the autonomous system one finds

- $P = P_0$: unstable de Sitter critical point. The critical value for the scalar curvature reads $R_0 = \sqrt{3}\mu^2$.
- $P = (-1/2, -1, -2, 0, -2/3)$: stable critical point. At the critical value, $H_0 = 0$.
- $P = (3(1+w)/2, -(5+3w), -2(5+3w), -3(4+3w), -(2+w))$: unstable critical point where $H_0 = 0$.
- $P = P_\Lambda$: unstable critical point. At the critical value one has $H_0^2 = (\Lambda/6)(1 + \sqrt{1 + 3\mu^4/4\Lambda^2})$.

$F = R + aR^2 + bP + cQ$ — (Starobinsky-like model). Here we have to assume $3a + b + c \neq 0$ otherwise the quadratic term becomes proportional to the Gauss-Bonnet invariant. For $0 \leq w \leq 1/3$, this model has only one critical point. In order to have a de Sitter solution, we have to introduce a cosmological constant Λ . We have in fact

- $P = P_0$: Minkowskian solution with $R_0 = 0$, which is stable if $3a + b + c > 0$.
- $P = P_\Lambda$: de Sitter critical point with $R_0 = 6\Lambda$, which is stable if $3a + b + c > 0$, in agreement with [53].

$F = R - d^2Q_3$ — This is the simplest toy model with the cubic invariant $Q_3 = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\mu\nu}R_{\gamma\delta}^{\mu\nu}$. For this model we have the following critical points:

- $P = P_0$: unstable de Sitter solution with $R_0 = 6/d$.
- $P = P_0$: stable Minkowskian solution with $R_0 = 0$.
- $P = (0.05, 0.60, -3.60, 0, -1.03)$: stable solution with $H_0 = 0$.
- $P = P_\Lambda$: this point exists, but not for any value of d and Λ . Also the value of H_0 and the stability depend on the parameters.

$F = R + aR^2 + bP + cQ - d^2Q_3$ — This is a generalisation of the previous two models. It may be motivated by the two-loop corrections in quantum gravity [56, 57]. The de Sitter critical points have been studied in Ref. [58]. The algebraic equations (2.21) are too complicated to be solved analytically, but it is easy to verify that there are at least the following solutions:

- $P = P_0$: de Sitter solution with $R_0 = 6/d$. This is stable if $3a + b + c + 3d > 0$.
- $P = P_0$: Minkowskian solution with $R_0 = 0$, which is stable if $3a + b + c > 0$.
- $P = P_\Lambda$: also in this case this point exists and is stable depending on the parameters (see [58]).

3 Conclusion

In this contribution, after a short review on the $F(R)$ case, we have presented the dynamical system approach which has permitted to arrive at a first order autonomous system of differential equations classically equivalent to the equations of motion for models of generalized modified gravity based on an arbitrary function $F(R, P, Q, Q_3, \dots)$, namely built up with all possible geometric invariant quantities of the FRW space-time. We have shown that, in the special case of $F(R)$ theories, the method gives rise to the well known results [22, 23, 24], but in principle it can be applied to the study of much more general cases.

As illustrations, we have discussed some simple models, for which a complete analytical analysis concerning the critical points has been carried out. However, in general, due to technical difficulties, a numerical analysis is required. Among the models investigated, we would like to remind that we were able to deal with one which involves a cubic invariant in the curvature tensor and, to our knowledge, this has never been considered before, and this shows the power of the present approach.

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